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A novel extended Kalman filter for a class of nonlinear systems*

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Abstract Estimation of the state variables of nonlinear systems is one of the fundamental and significant problems in control and signal processing. A new extended Kalman filtering approach for a class of nonlinear discrete-time systems in engineering is presented in this paper. In contrast to the celebrated extended Kalman filter (EKF), there is no linearization operation in the design procedure of the filter, and the parameters of the filter are obtained through minimizing a proper upper bound of the mean-square estimation error. Simulation results show that this filter can provide higher estimation precision than that provided by the EKF.

Keywords: nonlinear systems, state estimation, Kalman filtering.

Estimation of the state variables of a dynamic system through noisy measurements is one of the fundamental and significant problems in control and signal processing, and significant progress has been made in this area. In the 1940s, Wiener N., the founder of the modern statistical estimation theory, established Wiener filtering theory which solves the minimum variance estimation problem for stationary random processes. It was not until the late 1950s and early 1960s that Kalman filtering theory, a novel recursive filtering algorithm, was developed. It did not require the stationarity assumption, and has been widely used in many areas, such as aerospace and mechanical engineering. However, the Kalman filter is only applicable to linear systems. Since almost all the practical dynamic systems are nonlinear, Bucy and some other researchers were engaged in extending Kalman filtering theory to nonlinear systems in the following 10 years, and the most celebrated and widely used nonlinear filtering algorithm is the extended Kalman filter (EKF)[1], which is a suboptimal nonlinear filter. The key idea of the EKF is using the linearized dynamic model to calculate the covariance and gain matrices of the filter, thus the nonlinearity of the dynamic systems is stronger, the estimation precision provided by the EKF is worse. In the late 1990s, motivatied by the deficiencies of the EKF, Julier presented a new nonlinear filter named unscented Kalman filter (UKF) based on his work about the unscented transformation (UT)[2,3]. Though the UKF avoids the linearization operation, and can provide higher estimation precision than that of the EKF, selecting and propagating the sigma points will increase the computational cost in some cases. In the past few years, significant progress has been made in the robust Kalman filtering theory, and the key idea of robust Kalman filters is guaranteeing filtering precision through minimizing a proper upper bound of the mean-square estimation error[4—10], and this idea is also very constructive to the development of nonlinear filtering.

In this paper, we consider the state estimation problem of a class of nonlinear discrete-time systems whose state equation is nonlinear and measurement equation is linear. Based on the proper introduction of some assumptions and lemmas, a new extended Kalman filter (NEKF) is presented through minimizing an upper bound of the state mean square estimation error. A numerical simulation shows that the estimation precision provided by the NEKF is higher than that provided by the EKF.

1 Problem formulation

Consider the following nonlinear system defined on $k=0,1,\cdots,N-1$:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_k\boldsymbol{x}_k + \boldsymbol{B}_k\boldsymbol{w}_k + \boldsymbol{f}_k(\boldsymbol{x}_k), \tag{1a}$$

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$$y_k = C_k x_k + D_k v_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the measurement output, $w_k \in \mathbb{R}^m$ is the process noise, $v_k \in \mathbb{R}^q$ is the measurement noise. A_k, B_k, C_k, D_k are known matrices with proper dimensions, and $f_k(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying the following Assumption 1.

Assumption 1.

(i)
$$f_k(0) = 0, \tag{2}$$

(ii) There exists a matrix sequence $M_k \in \mathbb{R}^{n \times n}$, such that $\forall x_k^1, x_k^2 \in \mathbb{R}^n$

$$\| f_k(x_k^1) - f_k(x_k^2) \|_2 \leqslant \| M_k(x_k^1 - x_k^2) \|_2, \tag{3}$$

(iii) $\forall x_k \in \mathbb{R}^n$, there exists a known matrix $P(x_k)$ such that

$$f_k(x_k) = P_k(x_k) x_k. \tag{4}$$

Remark 1. If $x_k \neq 0$, then we can easily construct the matrix $P_k(x_k)$ satisfying Eq. (4). If $x_k = 0$, then since $f_k(0) = 0$, we can just let $P_k(x_k) = I$. Note that $P_k(x_k)$ can be singular.

Moreover, suppose w_k, v_k and the initial state x_0 have the following statistical properties:

$$\mathrm{E}\left\{\begin{bmatrix} w_k \\ v_k \\ x_0 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ 0 \\ \bar{x}_0 \end{bmatrix},$$

$$\mathrm{E}\left\{\begin{bmatrix} w_k \\ v_k \\ x_0 \end{bmatrix}\begin{bmatrix} w_j \\ v_j \\ x_0 \end{bmatrix}^{\mathrm{T}}\right\} = \begin{bmatrix} W_k\delta_{kj} & O & O \\ O & V_k\delta_{kj} & O \\ O & O & X_0 \end{bmatrix}, \tag{5}$$

where $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation operator, W_k, V_k and X_0 represent covariance matrices of the noises and the initial state, and δ_{kj} denotes the Kronecker delta function, which is equal to unity for $k = j$ and zero for other cases.

Now, we consider a filter of system (1) in the following form

$$\hat{x}_{k+1} = A_{ok}\hat{x}_k + K_{ok}(y_k - C_k\hat{x}_k) + f_k(\hat{x}_k), \tag{6}$$

where $k = 0, 1, \cdots, N-1$, $\hat{x}_k \in \mathbb{R}^n$ is the estimation value of the state, A_{ok} and K_{ok} are the filter parameters to be determined, and suppose $\hat{x}_0 = O$. From Eq. (4), we can see that $\forall \hat{x}_k \in \mathbb{R}^n$, there exists a known matrix $\hat{P}_k(\hat{x}_k) \in \mathbb{R}^{n \times n}$ such that

$$f_k(\hat{x}_k) = \hat{P}_k(\hat{x}_k)\hat{x}_k. \tag{7}$$

Moreover, the assumption of the relationship between $P_k = P_k(x_k)$ and $\hat{P}_k = \hat{P}(\hat{x}_k)$ is given as follows.

Assumption 2.

The relationship between P_k and $\hat{P}_k$ satisfies

$$P_k - \hat{P}_k = H_k F_k E_k, \tag{8}$$

where H_k and E_k are known matrices with proper dimensions, and $F_k \in R^{r \times s}$ is a norm-bounded matrix satisfying

$$F_k^{\mathrm{T}} F_k \leqslant I. \tag{9}$$

In the following, we shall derive the state-space model of the augmented system composed of the state equation (1a) and the nonlinear filter (6). Define a new state vector as

$$\tilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \tag{10}$$

and the state-space model of the augmented system can be represented as

$$\tilde{x}_{k+1} = \tilde{A}_k\tilde{x}_k + \tilde{B}_k\tilde{w}_k + \tilde{f}_k, \tag{11}$$

where

$$\tilde{A}_k = \begin{bmatrix} A_k & O \\ K_{ok}C_k & A_{ok} - K_{ok}C_k \end{bmatrix},$$

$$\tilde{B}_k = \begin{bmatrix} B_k & O \\ O & K_{ok}D_k \end{bmatrix},$$

$$\tilde{w}_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix}, \quad \tilde{f}_k = \begin{bmatrix} f_k(x_k) \\ f_k(\hat{x}_k) \end{bmatrix}.$$

Suppose the covariance matrix of the augmented system is

$$\tilde{\Sigma}_k = \mathrm{E}[\tilde{x}_k\tilde{x}_k^{\mathrm{T}}] = \mathrm{E}\left\{\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}^{\mathrm{T}}\right\}. \tag{12}$$

Then we have

$$\begin{aligned}\tilde{\Sigma}_{k+1} &= \mathrm{E}[(\tilde{A}_k\tilde{x}_k + \tilde{f}_k)(\tilde{A}_k\tilde{x}_k + \tilde{f}_k)^{\mathrm{T}}] + \tilde{B}_k\tilde{W}_k\tilde{B}_k^{\mathrm{T}} \\ &= (\tilde{A}_k + \tilde{P}_k + \tilde{H}_kF_k\tilde{E}_k) \\ &\quad \cdot \tilde{\Sigma}_k(\tilde{A}_k + \tilde{P}_k + \tilde{H}_kF_k\tilde{E}_k)^{\mathrm{T}} \\ &\quad + \tilde{B}_k\tilde{W}_k\tilde{B}_k^{\mathrm{T}},\end{aligned} \tag{13}$$

where

$$\tilde{H}_k = \begin{bmatrix} H_k \\ O \end{bmatrix}, \quad \tilde{E}_k = [E_k \quad O],$$

$$\tilde{W}_k = \begin{bmatrix} W_k & O \\ O & V_k \end{bmatrix}, \quad \tilde{P}_k = \begin{bmatrix} \hat{P}_k & O \\ O & \hat{P}_k \end{bmatrix},$$

and $\hat{P}_k = \hat{P}_k(\hat{x}_k)$ satisfies $f(\hat{x}_k) = \hat{P}_k\hat{x}_k$.

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In the following, we shall design a finite-horizon filter for structure (6), such that for all allowed uncertainties, there exists a sequence of positive definite matrices $\bar{\boldsymbol{\Sigma}}_k$ satisfying

$$[\boldsymbol{I} \quad -\boldsymbol{I}]\tilde{\boldsymbol{\Sigma}}_k[\boldsymbol{I} \quad -\boldsymbol{I}]^{\mathrm{T}} = \mathrm{E}[(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^{\mathrm{T}}] \leqslant \bar{\boldsymbol{\Sigma}}_k. \quad (14)$$

Because

$$\mathrm{tr}\{\mathrm{E}[(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^{\mathrm{T}}]\} = \mathrm{E}[(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^{\mathrm{T}}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)] \leqslant \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_k), \quad (15)$$

we shall minimize $\mathrm{tr}(\bar{\boldsymbol{\Sigma}}_k)$ and obtain an optimized filter eventually. Note that $\mathrm{tr}(\cdot)$ stands for the trace operator of a matrix.

2 Design of the filter

For the sake of the following discussion, two useful lemmas are introduced first.

Lemma 1.[7] Given matrices $\boldsymbol{A}$, $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ with compatible dimensions such that $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} \leqslant \boldsymbol{I}$, let $\boldsymbol{X}$ be a symmetrical positive-definite matrix and $\alpha > 0$ be an arbitrary positive constant such that $\alpha^{-1}\boldsymbol{I} - \boldsymbol{EXE}^{\mathrm{T}} > 0$. Then the following equality holds:

$$(\boldsymbol{A} + \boldsymbol{HFE})\boldsymbol{X}(\boldsymbol{A} + \boldsymbol{HFE})^{\mathrm{T}} \leqslant \boldsymbol{AXA}^{\mathrm{T}} + \boldsymbol{AXE}^{\mathrm{T}}(\alpha^{-1}\boldsymbol{I} - \boldsymbol{EXE}^{\mathrm{T}})^{-1}\boldsymbol{EXA}^{\mathrm{T}} + \alpha^{-1}\boldsymbol{HH}^{\mathrm{T}}. \quad (16)$$

Lemma 2.[9] Let $\boldsymbol{f}_k(\cdot): \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $0 \leqslant k \leqslant N$ be a sequence of matrix functions so that $\boldsymbol{f}_k(\boldsymbol{A}) = \boldsymbol{f}_k^{\mathrm{T}}(\boldsymbol{A})$, $\forall \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} > 0$, and $\boldsymbol{f}_k(\boldsymbol{B}) \geqslant \boldsymbol{f}_k(\boldsymbol{A})$, $\forall \boldsymbol{B} = \boldsymbol{B}^{\mathrm{T}} > \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} > 0$ hold. Let $\boldsymbol{g}_k(\cdot): \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $0 \leqslant k \leqslant N$ be a sequence of matrix functions so that $\boldsymbol{g}_k(\boldsymbol{A}) = \boldsymbol{g}_k^{\mathrm{T}}(\boldsymbol{A}) \geqslant \boldsymbol{f}_k(\boldsymbol{A})$, $\forall \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} > 0$. Thus the solution of the following difference equation

$$\boldsymbol{A}_{k+1} = \boldsymbol{f}_k(\boldsymbol{A}_k), \quad \boldsymbol{B}_{k+1} = \boldsymbol{g}_k(\boldsymbol{B}_k), \quad \boldsymbol{A}_0 = \boldsymbol{B}_0 > 0,$$

$\{\boldsymbol{A}_k\}_{0\leqslant k\leqslant N}$ and $\{\boldsymbol{B}_k\}_{0\leqslant k\leqslant N}$ $(0 \leqslant k \leqslant N)$ satisfies $\boldsymbol{A}_k \leqslant \boldsymbol{B}_k$.

Based on these two lemmas, we give the following theorem.

Theorem 1. If there exists a symmetrical positive definite matrix sequence $\boldsymbol{\Sigma}_k$ and a positive scalar sequence $\alpha_k (0 \leqslant k \leqslant N)$, such that

$$\alpha_k^{-1}\boldsymbol{I} - \tilde{\boldsymbol{E}}_k\boldsymbol{\Sigma}_k\tilde{\boldsymbol{E}}_k^{\mathrm{T}} > 0, \quad (17)$$

$$\boldsymbol{\Sigma}_{k+1} = \tilde{\boldsymbol{A}}_k\boldsymbol{\Sigma}_k\tilde{\boldsymbol{A}}_k^{\mathrm{T}} + \tilde{\boldsymbol{A}}_k\boldsymbol{\Sigma}_k\tilde{\boldsymbol{E}}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \tilde{\boldsymbol{E}}_k\boldsymbol{\Sigma}_k\tilde{\boldsymbol{E}}_k^{\mathrm{T}})^{-1}\tilde{\boldsymbol{E}}_k\boldsymbol{\Sigma}_k\tilde{\boldsymbol{A}}_k^{\mathrm{T}} + \alpha_k^{-1}\tilde{\boldsymbol{H}}_k\tilde{\boldsymbol{H}}_k^{\mathrm{T}} + \tilde{\boldsymbol{B}}_k\widetilde{\boldsymbol{W}}_k\tilde{\boldsymbol{B}}_k^{\mathrm{T}}, \quad (18)$$

$$\boldsymbol{\Sigma}_0 = \tilde{\boldsymbol{\Sigma}}_0 = \begin{bmatrix} \boldsymbol{X}_0 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}, \quad (19)$$

then

$$\tilde{\boldsymbol{\Sigma}}_k \leqslant \boldsymbol{\Sigma}_k, \quad (20)$$

where $\tilde{\boldsymbol{\Sigma}}_k$ satisfies Eq. (13).

Proof. If the matrix inequality (17) holds, then from Lemma 1, we have

$$\begin{aligned} \tilde{\boldsymbol{\Sigma}}_{k+1} &= (\tilde{\boldsymbol{A}}_k + \tilde{\boldsymbol{P}}_k + \tilde{\boldsymbol{H}}_k\boldsymbol{F}_k\tilde{\boldsymbol{E}}_k) \cdot \tilde{\boldsymbol{\Sigma}}_k(\tilde{\boldsymbol{A}}_k + \tilde{\boldsymbol{P}}_k + \tilde{\boldsymbol{H}}_k\boldsymbol{F}_k\tilde{\boldsymbol{E}}_k)^{\mathrm{T}} + \tilde{\boldsymbol{B}}_k\widetilde{\boldsymbol{W}}_k\tilde{\boldsymbol{B}}_k^{\mathrm{T}} \\ &\leqslant \tilde{\boldsymbol{A}}_k\tilde{\boldsymbol{\Sigma}}_k\tilde{\boldsymbol{A}}_k^{\mathrm{T}} + \tilde{\boldsymbol{A}}_k\tilde{\boldsymbol{\Sigma}}_k\tilde{\boldsymbol{E}}_k^{\mathrm{T}} \cdot (\alpha_k^{-1}\boldsymbol{I} - \tilde{\boldsymbol{E}}_k\tilde{\boldsymbol{\Sigma}}_k\tilde{\boldsymbol{E}}_k^{\mathrm{T}})^{-1}\tilde{\boldsymbol{E}}_k\tilde{\boldsymbol{\Sigma}}_k\tilde{\boldsymbol{A}}_k^{\mathrm{T}} \\ &\quad + \alpha_k^{-1}\tilde{\boldsymbol{H}}_k\tilde{\boldsymbol{H}}_k^{\mathrm{T}} + \tilde{\boldsymbol{B}}_k\widetilde{\boldsymbol{W}}_k\tilde{\boldsymbol{B}}_k^{\mathrm{T}}. \end{aligned}$$

Moreover, since

$$\boldsymbol{\Sigma}_0 = \tilde{\boldsymbol{\Sigma}}_0 = \begin{bmatrix} \boldsymbol{X}_0 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix},$$

then from Lemma 2, we can obtain that $\tilde{\boldsymbol{\Sigma}}_k \leqslant \boldsymbol{\Sigma}_k$, where $\boldsymbol{\Sigma}_k$ is defined by Eq. (18).

This completes the proof of Theorem 1.

Suppose

$$\bar{\boldsymbol{\Sigma}}_k = [\boldsymbol{I} \quad \boldsymbol{I}]\boldsymbol{\Sigma}_k[\boldsymbol{I} \quad \boldsymbol{I}]^{\mathrm{T}}, \quad (21)$$

then we can easily see that

$$\bar{\boldsymbol{\Sigma}}_k \geqslant [\boldsymbol{I} \quad -\boldsymbol{I}]\tilde{\boldsymbol{\Sigma}}_k[\boldsymbol{I} \quad -\boldsymbol{I}]^{\mathrm{T}} = \mathrm{E}[(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^{\mathrm{T}}], \quad (22)$$

and

$$\mathrm{tr}(\bar{\boldsymbol{\Sigma}}_k) \geqslant \mathrm{E}[(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^{\mathrm{T}}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)]. \quad (23)$$

In the following parts of this section, we will show how the filter parameters $\boldsymbol{A}_{ok}$ and $\boldsymbol{K}_{ok}$ are selected to minimize $\mathrm{tr}(\bar{\boldsymbol{\Sigma}}_k)$. Theorem 2 will summarize the approach to designing the filter to minimize $\mathrm{tr}(\bar{\boldsymbol{\Sigma}}_k)$.

Theorem 2. For a given positive scalar sequence α_k, inequality (17) holds. Then if

$$\boldsymbol{A}_{ok} = \boldsymbol{A}_k + (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k - \boldsymbol{K}_{ok}\boldsymbol{C}_k) \cdot \bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k, \quad (24)$$

and

$$\boldsymbol{K}_{ok} = (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1}, \quad (25)$$

then

$$\boldsymbol{\Sigma}_n = \begin{bmatrix} \boldsymbol{\Sigma}_{1,n} & \boldsymbol{\Sigma}_{2,n} \\ \boldsymbol{\Sigma}_{2,n} & \boldsymbol{\Sigma}_{2,n} \end{bmatrix}, \quad (n \in [0, N]), \quad (26)$$

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and tr $(\bar{\boldsymbol{\Sigma}}_k)$ is minimal, where

$$\boldsymbol{S}_k=\bar{\boldsymbol{\Sigma}}_k+\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k. \tag{27}$$

Moreover, the covariance matrices of the state and estimation error satisfy

$$\begin{aligned}\boldsymbol{\Sigma}_{1,k+1}=&(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)^{\mathrm{T}}\\&+\alpha_k^{-1}\boldsymbol{H}_k\boldsymbol{H}_k^{\mathrm{T}}+\boldsymbol{B}_k\boldsymbol{W}_k\boldsymbol{B}_k^{\mathrm{T}}+(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\\&\cdot\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\\&\cdot\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)^{\mathrm{T}},\end{aligned} \tag{28}$$

$$\begin{aligned}\bar{\boldsymbol{\Sigma}}_{k+1}=&(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{S}_k(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)^{\mathrm{T}}\\&+\boldsymbol{B}_k\boldsymbol{W}_k\boldsymbol{B}_k^{\mathrm{T}}+\alpha_k^{-1}\boldsymbol{H}_k\boldsymbol{H}_k^{\mathrm{T}}-(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\\&\cdot\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}+\boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1}\\&\cdot\boldsymbol{C}_k\boldsymbol{S}_k(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)^{\mathrm{T}},\end{aligned} \tag{29}$$

and $\boldsymbol{\Sigma}_{1,0}=\bar{\boldsymbol{\Sigma}}_0=\boldsymbol{X}_0$.

Proof. It is clear that if $n=0$, then Eq. (26) holds, i.e.

$$\boldsymbol{\Sigma}_0=\begin{bmatrix}\boldsymbol{\Sigma}_{1,0} & \boldsymbol{\Sigma}_{2,0}\\ \boldsymbol{\Sigma}_{2,0} & \boldsymbol{\Sigma}_{2,0}\end{bmatrix}=\begin{bmatrix}\boldsymbol{X}_0 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{bmatrix}.$$

Suppose Eq. (26) holds when $n=k$. In the following, we shall prove that Eq. (26) still holds when $n=k+1$. From Eq. (18), we have

$$\boldsymbol{\Sigma}_{k+1}=\begin{bmatrix}\boldsymbol{\Sigma}_{1,k+1} & \boldsymbol{\Sigma}_{12,k+1}\\ \boldsymbol{\Sigma}_{12,k+1}^{\mathrm{T}} & \boldsymbol{\Sigma}_{2,k+1}\end{bmatrix}, \tag{30}$$

where

$$\begin{aligned}\boldsymbol{\Sigma}_{1,k+1}=&(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)^{\mathrm{T}}\\&+\alpha_k^{-1}\boldsymbol{H}_k\boldsymbol{H}_k^{\mathrm{T}}+\boldsymbol{B}_k\boldsymbol{W}_k\boldsymbol{B}_k^{\mathrm{T}}+(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\\&\cdot\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\\&\cdot\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)^{\mathrm{T}},\end{aligned} \tag{31}$$

$$\begin{aligned}\boldsymbol{\Sigma}_{12,k+1}=&(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}\boldsymbol{C}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}}\\&+(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\\&\cdot\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{C}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}}+(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\\&\cdot\boldsymbol{\Sigma}_{2,k}(\hat{\boldsymbol{A}}_{ok}+\hat{\boldsymbol{P}}_k-\boldsymbol{K}_{ok}\boldsymbol{C}_k)^{\mathrm{T}}\\&+\alpha_k^{-1}\boldsymbol{H}_{1,k}\boldsymbol{H}_{2,k}^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}}+(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}\\&\cdot\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\boldsymbol{\Sigma}_{2,k}\\&\times(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k-\boldsymbol{K}_{ok}\boldsymbol{C}_k)^{\mathrm{T}},\end{aligned} \tag{32}$$

$$\begin{aligned}\boldsymbol{\Sigma}_{2,k+1}=&\boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{C}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}}+\boldsymbol{K}_{ok}\boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}}\\&+(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k}(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k)^{\mathrm{T}}\\&+[(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k}+\boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k]\\&\cdot\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\\&\times[(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k}+\boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k]^{\mathrm{T}}.\end{aligned} \tag{33}$$

From Eqs. (31)—(33), we have

$$\begin{aligned}\bar{\boldsymbol{\Sigma}}_{k+1}=&[\boldsymbol{I}\quad -\boldsymbol{I}]\boldsymbol{\Sigma}_{k+1}[\boldsymbol{I}\quad -\boldsymbol{I}]^{\mathrm{T}}\\=&(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k-\boldsymbol{K}_{ok}\boldsymbol{C}_k)\bar{\boldsymbol{\Sigma}}_k(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k-\boldsymbol{K}_{ok}\boldsymbol{C}_k)^{\mathrm{T}}\\&+(\boldsymbol{A}_{ok}-\boldsymbol{A}_k)\boldsymbol{\Sigma}_{2,k}(\boldsymbol{A}_{ok}-\boldsymbol{A}_k)^{\mathrm{T}}\\&+[(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k}+\boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k\\&-(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}]\\&\times\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\\&\times[(\boldsymbol{A}_{ok}+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k}+\boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k\\&-(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{1,k}]^{\mathrm{T}}+\alpha_k^{-1}\boldsymbol{H}_k\boldsymbol{H}_k^{\mathrm{T}}\\&+\boldsymbol{K}_{ok}\boldsymbol{V}_k\boldsymbol{K}_{ok}^{\mathrm{T}}+\boldsymbol{B}_k\boldsymbol{W}_k\boldsymbol{B}_k^{\mathrm{T}}.\end{aligned} \tag{34}$$

Moreover, we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{A}_{ok}}\\&=(\boldsymbol{A}_{ok}-\boldsymbol{A}_k)\\&\quad\cdot[\boldsymbol{I}+\boldsymbol{\Sigma}_{2,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k]\\&\quad-(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k-\boldsymbol{K}_{ok}\boldsymbol{C}_k)\\&\quad\cdot\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k,\end{aligned} \tag{35}$$

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2 \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{A}_{ok}^2}\\&=\boldsymbol{I}+\boldsymbol{\Sigma}_{2,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k>0,\end{aligned} \tag{36}$$

$$\begin{aligned}&\frac{1}{2}\frac{\partial \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{K}_{ok}}\\&=\boldsymbol{K}_{ok}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}+\boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]-(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}},\end{aligned} \tag{37}$$

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2 \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{K}_{ok}^2}\\&=\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}+\boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}>0.\end{aligned} \tag{38}$$

From Eqs. (35)—(38), if $\frac{\partial \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{A}_{ok}}=0$ and $\frac{\partial \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{K}_{ok}}=0$, then $\mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})$ is minimal.

From $\frac{\partial \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{A}_{ok}}=0$, we have

$$\begin{aligned}\boldsymbol{A}_{ok}=&\boldsymbol{A}_k+(\boldsymbol{A}_k+\hat{\boldsymbol{P}}_k-\boldsymbol{K}_{ok}\boldsymbol{C}_k)\\&\cdot\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\\&\times[\boldsymbol{I}+\boldsymbol{\Sigma}_{2,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I}-\boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k]^{-1}.\end{aligned} \tag{39}$$

Moreover, from the matrix inverse lemma, we have

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$$(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k \cdot [\boldsymbol{I} + \boldsymbol{\Sigma}_{2,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k]^{-1}$$
$$= [(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}}) + \boldsymbol{E}_k\boldsymbol{\Sigma}_{2,k}\boldsymbol{E}_k^{\mathrm{T}}]^{-1}\boldsymbol{E}_k$$
$$= (\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k. \tag{40}$$

Substituting (40) into (39), $\boldsymbol{A}_{ok}$ can be represented in a more concise form:

$$\boldsymbol{A}_{ok} = \boldsymbol{A}_k + (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k - \boldsymbol{K}_{ok}\boldsymbol{C}_k) \cdot \bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k. \tag{41}$$

From $\dfrac{\partial \mathrm{tr}(\bar{\boldsymbol{\Sigma}}_{k+1})}{\partial \boldsymbol{K}_{ok}} = 0$, we have

$$\boldsymbol{K}_{ok} = (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1}. \tag{42}$$

Let

$$\boldsymbol{M}_{1,k} = \boldsymbol{I} + \boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k,$$

and

$$\boldsymbol{M}_{2,k} = \boldsymbol{I} + \boldsymbol{\Sigma}_{2,k}\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k.$$

From Eq. (40), we can derive that

$$\boldsymbol{S}_k = \bar{\boldsymbol{\Sigma}}_k + \bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k$$
$$= [\boldsymbol{I} + \bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\boldsymbol{M}_{2,k}^{-1}]\bar{\boldsymbol{\Sigma}}_k$$
$$= [\boldsymbol{I} + (\boldsymbol{M}_{1,k} - \boldsymbol{M}_{2,k})\boldsymbol{M}_{2,k}^{-1}]\bar{\boldsymbol{\Sigma}}_k$$
$$= \boldsymbol{M}_{1,k}\boldsymbol{M}_{2,k}^{-1}\bar{\boldsymbol{\Sigma}}_k. \tag{43}$$

Substituting Eqs. (41), (42) and (43) into Eqs. (32) and (33) respectively, we have

$$\boldsymbol{\Sigma}_{12,k+1} = (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{M}_{1,k}\boldsymbol{\Sigma}_{1,k}\boldsymbol{C}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}} + (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{M}_{1k}\boldsymbol{\Sigma}_{2,k}(\boldsymbol{A}_{ok} - \boldsymbol{K}_{ok}\boldsymbol{C}_k)^{\mathrm{T}}$$
$$= (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1} \cdot \boldsymbol{C}_k\boldsymbol{S}_k(\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)^{\mathrm{T}} + \boldsymbol{A}_k\boldsymbol{M}_{1,k}\boldsymbol{M}_{2,k}^{-1}\boldsymbol{\Sigma}_{2,k}\boldsymbol{M}_{1,k}^{\mathrm{T}}\boldsymbol{A}_k^{\mathrm{T}}, \tag{44}$$

and

$$\boldsymbol{\Sigma}_{2,k+1} = \boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{C}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}} + (\boldsymbol{A}_{ok} + \hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k}(\boldsymbol{A}_{ok} + \hat{\boldsymbol{P}}_k)^{\mathrm{T}} + \boldsymbol{K}_{ok}\boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}\boldsymbol{K}_{ok}^{\mathrm{T}}$$
$$+ [(\boldsymbol{A}_{ok} + \hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k} + \boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k] \times \boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k^{\mathrm{T}})^{-1} \cdot \boldsymbol{E}_k[(\boldsymbol{A}_{ok} + \hat{\boldsymbol{P}}_k)\boldsymbol{\Sigma}_{2,k} + \boldsymbol{K}_{ok}\boldsymbol{C}_k\bar{\boldsymbol{\Sigma}}_k]^{\mathrm{T}}$$
$$= (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1} \cdot \boldsymbol{C}_k\boldsymbol{S}_k(\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)^{\mathrm{T}} + \boldsymbol{A}_k\boldsymbol{M}_{1,k}\boldsymbol{M}_{2,k}^{-1}\boldsymbol{\Sigma}_{2,k}\boldsymbol{M}_{1,k}^{\mathrm{T}}\boldsymbol{A}_k^{\mathrm{T}}. \tag{45}$$

Thus, from (44) and (45), we have

$$\boldsymbol{\Sigma}_{k+1} = \begin{bmatrix} \boldsymbol{\Sigma}_{1,k+1} & \boldsymbol{\Sigma}_{2,k+1} \\ \boldsymbol{\Sigma}_{2,k+1} & \boldsymbol{\Sigma}_{2,k+1} \end{bmatrix}.$$

Moreover, substituting Eqs. (41), (42) and (43) into Eq. (31), we have Eq. (28), and substituting Eqs. (28), (44) and (45) into Eq. (34), we can finally obtain that the covariance matrix of the estimation error $\boldsymbol{e}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k$ satisfies Eq. (29). This completes the proof of Theorem 2.

If the positive scalar sequence α_k is given, then the optimal filter is determined by Eqs. (24) and (25). Since α_k is also a parameter of the filter, how to choose α_k? The following theorem will solve the problem.

Theorem 3. If Eq. (17) holds, then α_k can be obtained from the following convex optimization problem

$$\min \mathrm{tr}(\boldsymbol{X}) \tag{46}$$

s.t.

$$\begin{bmatrix} [\boldsymbol{X} - (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k(\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)^{\mathrm{T}} - \boldsymbol{B}_k\boldsymbol{W}_k\boldsymbol{B}_k^{\mathrm{T}} - \alpha_k^{-1}\boldsymbol{H}_k\boldsymbol{H}_k^{\mathrm{T}}] & (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} \\ \boldsymbol{C}_k\boldsymbol{S}_k(\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)^{\mathrm{T}} & \boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}} \end{bmatrix} \geqslant 0, \tag{47}$$
$$\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}}, \quad 0 < \alpha_k \leqslant \| \boldsymbol{E}_k^{\mathrm{T}}\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k \|^{-1}.$$

Proof. It is clear that α_k can be obtained from the following convex optimization problem.

$$\min \mathrm{tr}(\boldsymbol{X})$$
$$\text{s.t. } \boldsymbol{X} \geqslant \bar{\boldsymbol{\Sigma}}_{k+1}, \quad \boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}},$$
$$0 < \alpha_k \leqslant \| \boldsymbol{E}_k^{\mathrm{T}}\boldsymbol{\Sigma}_{1,k}\boldsymbol{E}_k \|^{-1}.$$

By the use of Shur complement, $\boldsymbol{X} \geqslant \bar{\boldsymbol{\Sigma}}_{k+1}$ is equivalent to Eq. (47). Thus Theorem 3 has been proved.

Now, we summarize the novel extended Kalman filtering algorithm as follows:

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{A}_{ok}\hat{\boldsymbol{x}}_k + \boldsymbol{K}_{ok}(\boldsymbol{y}_k - \boldsymbol{C}_k\hat{\boldsymbol{x}}_k) + \boldsymbol{f}_k(\hat{\boldsymbol{x}}_k),$$
$$\boldsymbol{A}_{ok} = \boldsymbol{A}_k + (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k - \boldsymbol{K}_{ok}\boldsymbol{C}_k) \cdot \bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k,$$
$$\boldsymbol{K}_{ok} = (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1},$$
$$\boldsymbol{S}_k = \bar{\boldsymbol{\Sigma}}_k + \bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}}(\alpha_k^{-1}\boldsymbol{I} - \boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k\boldsymbol{E}_k^{\mathrm{T}})^{-1}\boldsymbol{E}_k\bar{\boldsymbol{\Sigma}}_k,$$
$$\boldsymbol{\Sigma}_{k+1} = (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k(\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)^{\mathrm{T}} + \boldsymbol{B}_k\boldsymbol{W}_k\boldsymbol{B}_k^{\mathrm{T}} + \alpha_k^{-1}\boldsymbol{H}_{1,k}\boldsymbol{H}_{1,k}^{\mathrm{T}} - (\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}}[\boldsymbol{C}_k\boldsymbol{S}_k\boldsymbol{C}_k^{\mathrm{T}} + \boldsymbol{D}_k\boldsymbol{V}_k\boldsymbol{D}_k^{\mathrm{T}}]^{-1} \cdot \boldsymbol{C}_k\boldsymbol{S}_k(\boldsymbol{A}_k + \hat{\boldsymbol{P}}_k)^{\mathrm{T}},$$

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where $\bar{\boldsymbol{\Sigma}}_0 = \boldsymbol{X}_0$, α_k is determined by Theorem 3, and $\hat{\boldsymbol{P}}_k = \hat{\boldsymbol{P}}_k(\hat{\boldsymbol{x}}_k)$ satisfies $\boldsymbol{f}(\hat{\boldsymbol{x}}_k) = \hat{\boldsymbol{P}}_k \hat{\boldsymbol{x}}_k$.

3 A numerical simulation

In order to evaluate the performance of the suboptimal filter presented in the former part of this paper, a numerical simulation is given as follows. Consider the following nonlinear discrete-time system:

$$\begin{cases} \boldsymbol{x}_{k+1} = \begin{bmatrix} 0.01 & -0.5 \\ 1 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{w}_k \\ \qquad + \lambda \begin{bmatrix} (|x_{1,k}|)^{\frac{2}{3}} \\ (|x_{2,k}|)^{\frac{2}{3}} \end{bmatrix}, \\ \boldsymbol{y}_k = [-100 \quad 10]\boldsymbol{x}_k + [0 \quad 1]\boldsymbol{v}_k \end{cases} \tag{48}$$

where $\lambda > 0$ is a positive scalar. Let

$$\boldsymbol{W}_k = 0.01\boldsymbol{I}_2, \quad \boldsymbol{V}_k = 0.01\boldsymbol{I}_2, \quad \boldsymbol{X}_0 = 0.01\boldsymbol{I}_2,$$
$$\boldsymbol{H}_k = 0.1\boldsymbol{I}_2, \quad \boldsymbol{E}_k = [(0.5)^k + 0.5]\boldsymbol{I}_2,$$

and $\bar{x}_0 = [10 \quad 10]^{\mathrm{T}}$, $k \in [0, N]$. It is clear that we can choose $\hat{\boldsymbol{P}}_k(\hat{\boldsymbol{x}}_k)$ such that

$$\hat{\boldsymbol{P}}_k(\hat{\boldsymbol{x}}_k) = \begin{cases} \begin{bmatrix} \dfrac{(|\hat{x}_{1,k}|)^{\frac{2}{3}}}{2\hat{x}_{1,k}} & 0 \\ 0 & \dfrac{(|\hat{x}_{2,k}|)^{\frac{2}{3}}}{2\hat{x}_{2,k}} \end{bmatrix} & \hat{x}_{1,k} \neq 0, \hat{x}_{2,k} \neq 0 \\ \begin{bmatrix} \dfrac{(|\hat{x}_{1,k}|)^{\frac{2}{3}}}{2\hat{x}_{1,k}} & 0 \\ \dfrac{(|\hat{x}_{2,k}|)^{\frac{2}{3}}}{2\hat{x}_{1,k}} & 0 \end{bmatrix} & \hat{x}_{1,k} \neq 0, \hat{x}_{2,k} = 0 \\ \begin{bmatrix} 0 & \dfrac{(|\hat{x}_{1,k}|)^{\frac{2}{3}}}{2\hat{x}_{2,k}} \\ 0 & \dfrac{(|\hat{x}_{2,k}|)^{\frac{2}{3}}}{2\hat{x}_{2,k}} \end{bmatrix} & \hat{x}_{1,k} = 0, \hat{x}_{2,k} \neq 0 \\ \boldsymbol{I}_2 & \hat{x}_{1,k} = 0, \hat{x}_{2,k} = 0. \end{cases} \tag{49}$$

Define the mean square estimation error of the kth step as

$$\mathrm{MSE}_k = \frac{1}{k} \sum_{i=0}^{k} (\boldsymbol{x}_i - \hat{\boldsymbol{x}}_i)^{\mathrm{T}} (\boldsymbol{x}_i - \hat{\boldsymbol{x}}_i). \tag{50}$$

The estimation values of the system state using the NEKF and the EKF with $\lambda = 0.1$ are illustrated in Fig. 1. The mean-square errors during the estimation procedure and the values of α_k are illustrated in Fig. 2. The estimation values of the system state using the NEKF and the EKF $\lambda = 0.5$ are illustrated in Fig. 3, and the mean-square errors during the estimation procedure and the values of α_k are illustrated in Fig. 4.

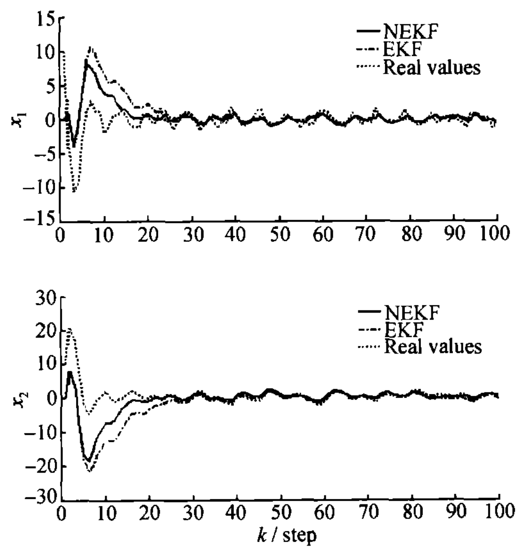

Fig. 1. State estimation values with $\lambda = 0.1$.

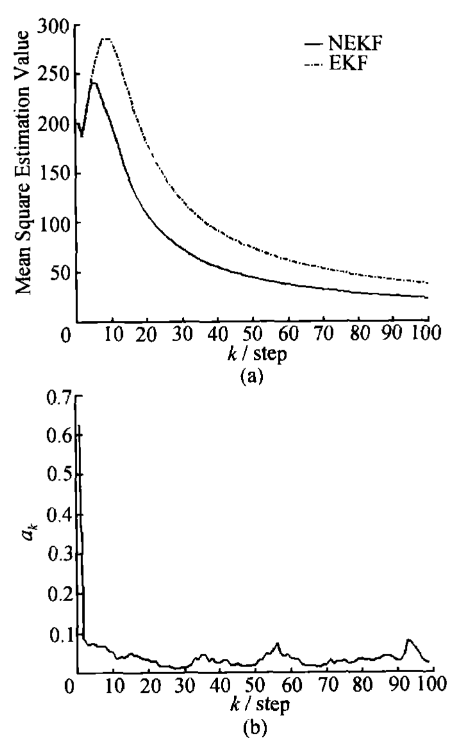

Fig. 2. Mean-square error with $\lambda = 0.1$ (a) and the values of α_k with $\lambda = 0.1$ (b).

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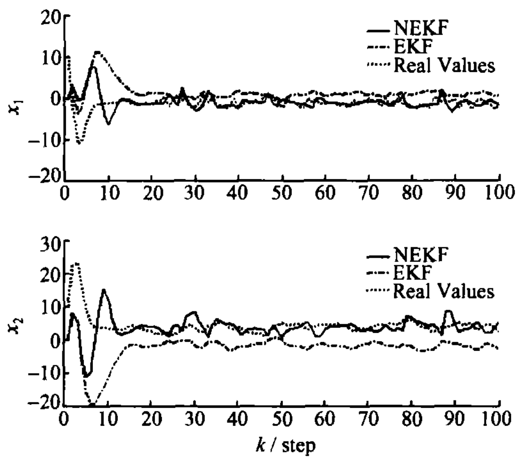

Fig. 3. State estimation values with $\lambda = 0.5$.

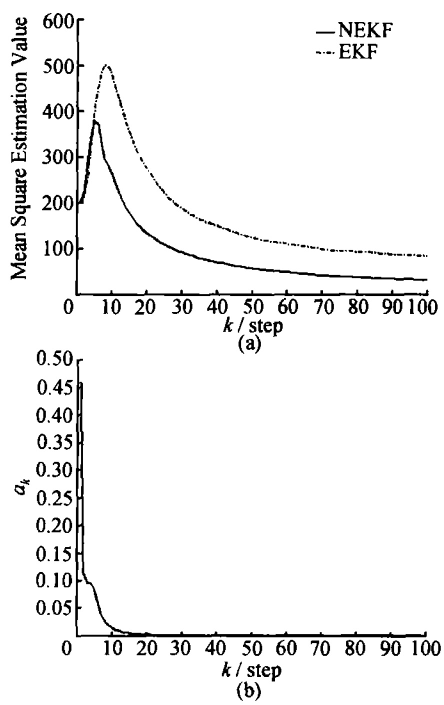

Fig. 4. Mean-square error with $\lambda = 0.5$ (a) and the values of a_k with $\lambda = 0.5$ (b).

From the simulation results, we can see that the estimation precision of the NEKF is higher than that of the EKF, and from Figs. 1 and 3, we can easily see that the nonlinearity in the dynamic model is stronger, and the difference between the estimation precisions of the two filters is bigger, mainly because the parameters of the filter are calculated using the original nonlinear dynamic model in the design procedure of the NEKF. Thus, when the nonlinearity of a system is strong, we had better choose the NEKF to estimate the state of the system, and the simulation results show that the novel extended Kalman filter is feasible.

4 Conclusion

A novel extended Kalman filtering approach for a class of nonlinear discrete-time systems with nonlinear state equation and linear measurement equation has been presented in this paper. Because the design procedure of the new suboptimal nonlinear filter avoids linearization operation, the estimation precision provided by this new filter is higher than that provided by the celebrated EKF especially when the nonlinearity of dynamical systems is stronger. Simulation results show the effectiveness of the suboptimal filter. Furthermore, the filtering approach can be utilized in areas such as navigation and attitude determination of spacecrafts.

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